**Formal Properties of Green’s Functions**

Might have noticed that calculating the time-evolution of wavefunctions is hard. Calculating the time evolution of operators is easier, at least insofar as we saw for the harmonic oscillator. GF’s are expectations of the time evolution of operator combinations. The time evolution of these guys is generally easier to calculate (especially as the number of particles increases beyond one – see multiple particles folders, quantum field theory, and statistical mechanics, and condensed matter). And they can be just as informative. Many physical quantities of interest can be put in terms of Green’s Functions.

Here are the 6 common green’s functions. For the Heisenberg picture operators, I think we must use the typical U(t,t0) = U0(t,t0)S(t,t0), as I don’t think the mixed phase convention time-development operator is compatible with the adiabatic theorem, which will be used later (see QM/Time-Dependent/Adiabatic Perturbation). Note the time-ordering operator orders things later on left. Note that the t argument is always with the A, and t′ always with the B. And the lesser/advanced GF are the only ones with the sign change.



These are the causal, anticausal, greater, lesser, retarded, and advanced GF’s respectively. These GF’s also implicitly carry with them exponential convergence factors that come in with the θ functions: θ(t) → θ(t)e-ηt, θ(-t) → θ(-t)eηt, where η = 0+ is an infinitesimal positive number, eventually taken to zero. Also, I think even the >,< GF’s carry these factors, and should be technically defined as, for instance,

G>(t,t´) = -iθ(t-t´)<AH(t)BH(t´)> - iθ(t´-t)<AH(t)BH(t´)>

→

G>(t,t´) = -iθ(t-t´)e-η(t-t´)<AH(t)BH(t´)> - iθ(t´-t)eη(t-t´)<AH(t)BH(t´)>.

Anyway, since this limit gives us 1 (unless t = ∞, or t´ = -∞), we will just leave it off, to save space, unless necessary. The exponential convergence factors are there, I think, because of the use of the adiabatic theorem:|GS> = Sη(t0,-∞)|GS0>, where the η is there to indicate exponentially slowly turning on the perturbation (see Quantum Mechanics/Time-Development/Adiabatic Perturbation). This is where eηt factors would show up in calculations. And I believe this convergence factor, initially in S, gets transferred to the GF’s and so effectively becomes part of their definition for any calculation involving times in infinite past or future (where there is a e-ηt term in the adiabatic theorem). For finite times, the exponential factors are superfluous, as their limit will always gives us 1.

An alternate notation: GC = G--, GAC = G++, G> = G+-, G< = G-+ is also used. I have contemplated putting a complex time-ordered GF in here GC\*(-iτ,-iτ´). But it doesn’t seem to be well-defined in general – it would seem to blow up for large |τ-τ´| if the background state is not the vacuum. If we are using the vacuum, then it will decay for positive τ–τ´, but blow up for negative. But if τ-τ´ can never be negative, then there is only one option for the time-ordering: must have AH(τ) before BH(τ´), which makes the time-ordering deal a little superfluous. Also we can’t in general take the FT, unless we’re working in vacuum, and can’t get spectral function either therefore. So I think it’s best to just abandom the concept here; it works for the thermal averaging deal because of the periodicity property that it holds, which allows us to have positive/negative times, meaningful time-ordering, and a FT + spectral function.

In order for the Wick expansion, that we’ll use later, to hold, we must have that the background expectation state be an eigenstate of some bilinear H0 (or combination of such) |Ω0>



In many body case, we need |Ω0> to be ‘homogeneous’, meaning presumably that occupation #’s ~ 1/N. But here in single particle land this wouldn’t make sense. It might be that Wick’s theorem will only work, then, for |GS0> etc. The operators A, B can be anything, and may evolve in time according to some Hamiltonian,



where the potential may contain time-independent and dependent parts.

**Algebraic relationships between the various Green’s functions**

The green’s functions aren’t independent. Some algebraic relations between the green’s functions are as follows.



So we have,



And consider:



So we have:



And we can also say:



Also consider,



and so we have,



and from the top relationship,



and so we have,



or could say,



so,



**Symmetries of GF’s**

Consider a GF evaluated against some state |Ω0>. And presume time-development against a constant H. Constant H would mean energy is conserved. Then for instance our GF should depend only on t-t´, i.e., be time-translationally invariant.



Now if |Ω0> is an eigenstate of H, then:



But note that if we evaluated against states that weren’t eigenstates of H (like ones that were eigenstates of H0, rather), then this would not be time-translationally invariant.

**Green’s Functions in terms of the Spectral Function**

The following discussion will assume that we’re using the expectation basis, |Ω0>, which is an arbitrary state. We assume the operators develop according to a **time-independent** H for the spectral properties to hold. Now let’s look at the Fourier transform of the various green’s functions.

**Retarded Green’s Function**

Consider that we have the retarded Green’s function GR(t-tʹ). We can take the Fourier transform (reinserting the exponential convergence factor of necessity). Note there is implicit summation over m and n indices (third line and down)



And finally this can be written as:



where we define a spectral function A(x).



Let’s pause to observe that the poles of G are the *system excitations*. But they are modulated by the presence of Ω0. So only excitations that have non-zero overlap with |Ω0> will be present.

**Advanced Green’s function**

If we carry out the same procedure on the advanced Green’s function, we would evaluate the Fourier transform, and obtain:



**Greater Green’s function**

Now let’s consider the Fourier transform of the greater Green’s function,



The first two terms will give us the spectral function, and the second can be written as f(ω)A(ω), where we define f(ω) via the equality.



**Lesser Green’s function**

And we also have for the lesser GF.



We can put this in terms of the spectral function as well, and we’ll find:



Lesser GF has less terms, ha!

**Causal Green’s function**

And we also have for the causal GF.



We can put this in terms of the spectral function as well, and we’ll find, most simply from the algebraic relations at the top of the page



**Anti-causal Green’s function**

And we also have for the causal GF.



We can put this in terms of the spectral function as well, and we’ll find, most simply from the algebraic relations at the top of the page



**Properties of the Spectral Function**

From the forms of the less/greater, and retarded/advanced GF’s, we can see that the spectral function can be constructed by:



If A(ω) is real, then GA = (GR)\* we can say:



Moreover, we saw the Spectral Function was explicitly given by:



So it clearly has poles (infinities) at the excitations of the system (since |m>, |n> are the eigenstates, and so En – Em would be the excitations). But these are modulated by the overlap of the excitation states with |Ω0> and the operators A and B. The reality of A(x) seems to hinge on a lot of things. So if A and B are Hermitian conjugates of each other (or themselves), and if |Ω0> is an eigenstate of H, then A(x) should be real since:



And then its complex conjugate would be:



So there.

**Kramers - Kronig relations obeyed by GF’s and A(ω)**

So let’s analytically continue the GF’s so that ω can be a complex variable. We might just define:



where ω is a complex variable, again. And GR(ω∈ℝ) = G(ω+iδ) and GA(ω∈ℝ) = G(ω-iδ). This function would be analytic in the u.h.p., and in the l.h.p., but would have a branch cut along the real axis. All functions analytic in one half plane may be shown to obey what are called Kramers – Kronig relations, which relate the real and imaginary parts of the function (along the branch cut?). So consider (ω implicitly real here) …



where in the last we **presume A(ω) is real** so that A(ω) = ImGR(ω)/π. Then we have that,



and



I can just remember that the real and imaginary parts are related via a Hilbert Transform, the signs, I suppose, aren’t too critical. So our Kramers – Kronig relations, relating the real and imaginary parts of our analytic function, G(ω), just above the real line reads:



The advanced Green’s function would obey a similar relation. In fact, these relationships follow more generally than we took advantage of above (where we required the spectral function to be real). We’ll apply the more general consideration to the spectral function itself for instance. Suppose that it also is analytic in the u.h.p. and that it goes to 0 quickly enough. Then by Cauchy’s theorem,



Then let z = x + i0+, for example. Then we have:



and so finally,



where the integral is to be interpreted in the p.v. sense. Now we can determine KK relations.



and so



and so,



These last two relations come directly from the properties of the H transform.

**Expectations from Green’s functions**

Consider a non-harmonic spring Hamiltonian, in an external time-dependent field:



Suppose we are in some initial state |Ω0>. We let it evolve according to H, and then want to calculate the average spring potential energy as a function of time thereafter. Then we’d calculate:



We can extract this from a GF, say GFC­­ (or most of the others, basically, but this is usually the easiest to calculate):



We’d have:



We could get KE too, using vH(t) = H(t),



Taking advantage of the differential equation for GC(t,t’), which we’ll discuss below, it looks like we could get <H> too. I won’t go into it I guess. But once we have that, then we could do <H – KE>, and then by taking derivatives w/r to h or λ, we could get <x> or <x3>. So just about everything we’d want to calculate is encapsulated within the GF.

**Expectations from the spectral function**

From the definition of G<, we have:



We also have,



This property actually follows from the definition of A in terms of GR and GA. So it is completely general. For instance,



Let’s consider calculating <AB>. We can clearly find the appropriate linear combination of the two above.

